SOLUTIONS TO THE EQUATIONS OF A COMPRESSIBLE LAMINAR BOUNDARY LAYER AT A PLATE WITH SUDDENLY CHANGING BOUNDARY CONDITIONS

V. I. Eliseev

UDC 532.526.2

A method is proposed for constructing the solutions to the equations of a compressible boundary layer at a plate with suddenly changing boundary conditions, when u ~ T and Pr = 1.

A sudden change in the boundary conditions of a boundary layer reveals a region where the fundamental premises of the theory of the laminar boundary layer cease to be valid. The characteristic dimension of this region is commensurable with the thickness of the boundary layer and its effect on the mainstream is revealed when quantities of second-order smallness are taken into account, making it feasible to analyze such problems by the methods used in the theory of the boundary layer for obtaining solutions for the contiguous regions, which can then be adjoined according to one or another principle.

Here the author uses the method of outer and inner expansions [1] for constructing the sought solution. It will be assumed that region 2 (Fig. 1) where the boundary conditions differ from the original ones lies inside region 1. The idea of subdividing the entire boundary layer into an outer and an inner region, with the inner one treated as a new boundary layer, was already conceived and used earlier in [2-9].

In order to apply the method of adjoint asymptotic expansions, we introduce a parameter ε which as a small parameter would be used for obtaining the outer and the inner coordinates. Let δ_1 be a quantity characterizing the thickness of the boundary layer at any section x and let δ_2 be a quantity characterizing the thickness of region 2. Considering that region 2 begins at a certain point x_* and then gradually expands in the downstream direction until at $x \to \infty$ it merges into the entire boundary layer, we conclude that $\varepsilon = \delta_2/\delta_1$ varies from 0 to 1 as x runs from x_* to infinity. Such an interval of ε values makes this parameter suitable for adjoining the outer and the inner expansions.

We write the fundamental equations for the simplest case, namely for $\mu \sim T$ and Pr = 1, in dimensionless form:

$$\frac{\partial\Psi}{\partial\eta}\frac{\partial^2\Psi}{\partial\xi\partial\eta}-\frac{\partial\Psi}{\partial\xi}\frac{\partial^2\Psi}{\partial\eta^2}=\frac{\partial^3\Psi}{\partial\eta^3},$$
$$\frac{\partial\Psi}{\partial\eta}\frac{\partial^2 Q}{\partial\xi\partial\eta}-\frac{\partial\Psi}{\partial\xi}\frac{\partial^2 Q}{\partial\eta^2}=\frac{\partial^3 Q}{\partial\eta^3}+(k-1)\mathrm{M}_\infty^2\left(\frac{\partial^2\Psi}{\partial\eta^2}\right)^2, \tag{1}$$

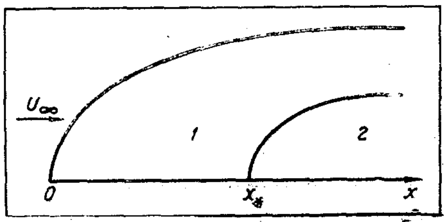

Fig. 1. Schematic diagram of the flow in a boundary layer: 1) outer region; 2) inner region.

where ξ, η are Dorodnitsyn variables [10], ψ is the flow function for the boundary layer, $Q = \int_0^\eta \tau d\eta$, and τ is the dimensionless temperature.

The solution for the outer and the inner region will be represented as

$$\Psi = \Psi_1 = \delta_1 [F_0(n) + \varepsilon F_1(n) + \varepsilon^2 F_2(n) + \ldots],$$

Three-Hundredth-Anniversary-of-the-Union-between-Ukraine-and-Russia State University, Dnepropetrovsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 24, No. 3, pp. 445-452, March, 1973. Original article submitted April 7, 1972.

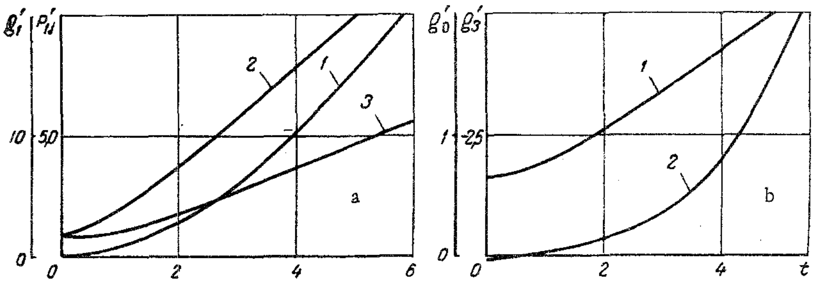

Fig. 2. Graphs of functions (a) g_1' (1), p_{11}' (2), p_{12}' (3), (b) g_0' (1), g_3' (2).

$$Q = Q_1 = \delta_1 [R_0(n) + \varepsilon R_1(n) + \varepsilon^2 R_2(n) + \dots], \quad n = \eta/\delta_1, \tag{2}$$

$$\begin{gathered} \Psi = \Psi_2 = \delta_1 \varepsilon^2 [G_0(N) + \varepsilon G_1(N) + \varepsilon^2 G_2(N) + \dots], \\ Q = Q_2 = \delta_1 \varepsilon [P_0(N) + \varepsilon P_1(N) + \varepsilon^2 P_2(N) + \dots], \\ N = \eta/\delta_2. \end{gathered} \tag{3}$$

respectively. In order that a solution in this form satisfy Eqs. (1), it is necessary that the following conditions be satisfied for δ_1 and δ_2:

$$\begin{gathered} \delta_1' = a\delta_1^{-1}(1 + b_1\varepsilon + b_2\varepsilon^2 + \dots), \\ \delta_2' = b\delta_1^{k_0}\delta_2^{p}(1 + c_1\varepsilon + c_2\varepsilon^2 + \dots). \end{gathered} \tag{4}$$

Here b_j and c_j are correction factors accounting for any perturbations which may occur as a result of interaction between regions 1 and 2 behind the point $\xi = 1$. With the aid of expressions (4), we obtain the derivative of ε with respect to ξ:

$$\varepsilon' = a\delta_1^{-2}\varepsilon \left[\varepsilon^{p-1}(1 + c_1\varepsilon + c_2\varepsilon^2 + \dots)\frac{b}{a} - (1 + b_1\varepsilon + b_2\varepsilon^2 + \dots)\right],$$

where $p = -(1 + k_0)$. This relation yields integral power exponents in the expression for ε'. Eliminating ξ from (4), we have

$$\frac{d\delta_2}{d\delta_1} = \frac{b}{a}\varepsilon^p \frac{1 + c_1\varepsilon + c_2\varepsilon^2 + \dots}{1 + b_1\varepsilon + b_2\varepsilon^2 + \dots}.$$

The set of factors b_j and c_j cannot be determined uniquely, inasmuch as we do not have the sufficient number of equations. Considering this, and also noting that $\delta_2'|_{\xi=1} > 0$, i.e., $d\delta_2/d\delta_1|_{\delta_1(1)} > 0$ and $\delta_2 \to \delta_1$ or $d\delta_2/d\delta_1 \to 1$ at $\xi \to \infty$, we relate factors b_j and c_j in the simplest manner: $b = a$ and $c_j = b_j$. Now p will be determined after the inner expansions (3) have been inserted into Eqs. (1), with the stipulation that the viscous and the dynamic terms in the equations for G_0 and P_0 be retained. This requirement is satisfied with $p = -2$. It is easy then to obtain the expressions

$$\begin{gathered} \varepsilon' = a\delta_1^{-2}\varepsilon^{-2}(1 - \varepsilon^3)(1 + b_1\varepsilon + b_2\varepsilon^2 + \dots), \\ \delta_1 = \delta_1(1)(1 - \varepsilon^3)^{1/3}, \\ \delta_2 = \left[\delta_1^3 - \delta_1^3(1)\right]^{1/3}. \end{gathered} \tag{5}$$

We write the equations for both the outer and the inner region, after expansions (2) and (3) have been inserted into the equations of the boundary layer:

for the outer region

$$\begin{gathered} F_0'F_1' - F_0''F_1 = 0, \\ F_0'F_2' - F_0''F_2 = -\frac{1}{2}(F_1'^2 - F_1F_1''), \end{gathered} \tag{6}$$

$$F_0'F_3' - F_0''F_3 = -\frac{1}{3}\left[2(F_1'F_2' - F_1''F_2) + (F_1'F_2' - F_1F_2'') + 2b_1(F_0'F_2' - F_0''F_2) + b_1(F_1'^2 - F_1F_1'') + \frac{1}{3a}[F_0''' + aF_0F_0''\right],$$

$$\begin{gathered} F_0'R_1' = F_1R_0'', \\ F_0'R_2' = -\frac{1}{2}(F_1'R_1' - F_1R_1'' - 2F_2R_0''), \end{gathered} \tag{7}$$

$$F_0'R_3' = -\frac{1}{3}(2F_1'R_2' + F_2'R_1' - F_1R_2'' - 2F_2R_1'' - 3F_3R_0'') + \frac{1}{3a}\left[R_0''' + F_0R_0'' + (k-1)M_\infty^2 F_0''^2\right],$$

for the inner region

$$G_0''' = a(G_0'^2 - 2G_0G_0''),$$
$$G_1''' = a\left[3G_0'G_1' - 2G_0G_1'' - 3G_0''G_1 + b_1(G_0'^2 - 2G_0G_0'')\right],$$
$$G_2''' = a\left[4G_0'G_2' - 2G_0G_2'' - 4G_0''G_2 + 2G_1'^2 - 3G_1G_1'' + b_1(3G_0'G_1' - 2G_0G_1'' - 3G_0''G_1) + b_2(G_0'^2 - 2G_0G_0'')\right], \tag{8}$$
$$P_0''' = -a2G_0P_0'',$$
$$P_1''' = -a(2G_0P_1'' - G_0'P_1' + 3G_1P_0'' + b_12G_0P_0''),$$
$$P_2''' = -a\left[2G_0P_2'' - 2G_0'P_2' - G_1'P_1' + 3G_1P_1'' + 4G_2P_0''\right.$$
$$\left. - b_1(G_0'P_1' - 2G_0P_1'' - 3G_1P_0'') + b_22G_0P_0''\right] - (k-1)M_\infty^2 G_0''^2. \tag{9}$$

The last terms inside the square brackets in the third equation of system (6) and of system (7) are equal to zero, because functions F_0 and R_0 are solutions to the equations of the boundary layer at a semiinfinite flat plate. The subsequent solutions F_j and R_j can be found by simply integrating Eqs. (6) and (7), which will yield:

$$F_1 = k_1F_0',$$
$$F_2 = \frac{1}{2!}(k_1^2F_0'' + 2k_2F_0'), \tag{10}$$
$$F_3 = \frac{1}{3!}(k_1^3F_0''' + 6k_1k_2F_0'' + 6k_3F_0'),$$

$$R_1 = k_1R_0' + s_1,$$
$$R_2 = \frac{1}{2!}(k_1^2R_0'' + 2k_2R_0') + s_2, \tag{11}$$
$$R_3 = \frac{1}{3!}(k_1^3R_0''' + 6k_1k_2R_0'' + 6k_3R_0') + s_3.$$

The integration constants k_j and s_j are determined from the condition of adjoint solutions which, according to the rule of adjoint asymptotic expansions, can be written as:

$$G_j|_{N\to\infty} \to \sum_{m=0}^{j+1} \frac{1}{(j+2-m)!} F_m^{(j+2-m)}(0)\,N^{j+2-m},$$
$$P_j|_{N\to\infty} \to \sum_{m=0}^{j} \frac{1}{(j+1-m)!} R_m^{(j+1-m)}(0)\,N^{j+1-m}. \tag{12}$$

Equations (8) and (9) admit the following asymptotic form of functions G_j and P_j at $N \to \infty$:

$$G_j = \sum_{m=0}^{j+2} a_{jm}N^m, \quad P_j = \sum_{m=0}^{j+1} d_{jm}N^m, \tag{13}$$

where the coefficients a_{jm} and d_{jm} can be found from expressions (12), with the possibility of ascertaining that $a_{j0} = F_{j+2}(0)$ and $d_{j0} = R_{j+1}(0)$. These last expressions indicate that all rules of adjoint solutions have been obeyed at any b_j.

Boundary Layer with Injection or Suction in Region 2. The boundary condition for function ψ at $N = 0$ will be stipulated as:

$$\Psi = \delta_1\varepsilon^3(A_1 + \varepsilon A_2 + \ldots). \tag{14}$$

Since it follows from (5) that $\xi = 1 + \varepsilon^3(v_0 + \varepsilon v_1 + \ldots)$, hence the condition that the first term in expansion (14) is proportional to ε^3 corresponds to the boundedness of the normal injection or suction velocity in the vicinity of the point $\xi = 1$. When $A_j = 0$, the inner region vanishes and the outer region extends to the plate surface. Expression (14) indicates that the boundary conditions for the first term in the inner extension of G_0 are homogeneous at $N = 0$ and that then

$$G_0 = \frac{1}{2}F_0''(0)N^2, \tag{15}$$

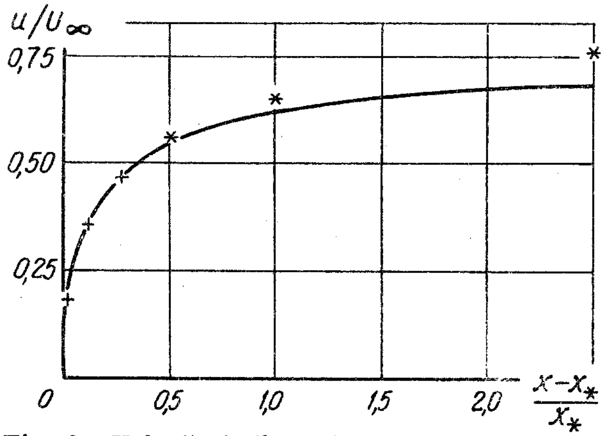

Fig. 3. Velocity in the wake along its axis: solid curve represents formula (33), dots represent values according to [11].

which yields $k_1 = 0$. The solution to the second equation in (8) will be written as

$$G_1 = G_{10} + C_1 G_{11}, \tag{16}$$

where $G_{10} = -1/3! \cdot a3F_0''(0)A_1N^3 + A_1$ and G_{11} is determined from the following boundary conditions: $G_{11}(0) = G_{11}'(0) = 0$ and $G_{11}''(0) = 1$. With the substitution $G_{11} = [1/2 \cdot aF_0''(0)]^{-2/3} g_{11}\{[1/2 \cdot aF_0''(0)]^{1/3}N\}$ we obtain for g_{11}:

$$g_{11}''' + 2t^2 g_{11}'' - 6t g_{11}' + 6g_{11} = 0, \quad t = \left[\frac{1}{2}\, aF_0''(0)\right]^{1/3} N,$$
$$g_{11}(0) = g_{11}'(0) = 0, \quad g_{11}''(0) = 1. \tag{17}$$

Equation (17) was integrated numerically and its solution is shown in Fig. 2a. Here C_1 and k_2 have been determined from the boundary conditions at infinity:

$$C_1 = 5.074 \left[\frac{1}{2}\, aF_0''(0)\right]^{2/3} A_1, \quad k_2 = 0.329 A_1.$$

In order to obtain a solution for Q_2, we express the boundary condition for this function at $N = 0$ as

$$Q_2' = B_0 + \varepsilon B_1 + \varepsilon^2 B_2,$$

and the solution in the first approximation will be

$$P_0 = B_0 N + P_0''(0) \int_0^N \int_0^N \exp\left[-2a \int_0^N G_0 dN\right] dN dN. \tag{18}$$

It is evident from (18) that, if $P_0''(0) \neq 0$, i.e., if a temperature jump occurs at the point $\xi = 1$, then the magnitude of the thermal flux in the vicinity becomes infinite. We next consider the case where $P_0''(0) = 0$. The solution to the heat problem of the boundary layer at a semiinfinite plate will be written for two cases, namely a thermally insulated plate and an isothermal plate [10]:

$$\tau_0 = 1 + \frac{k-1}{2} M_\infty^2 - \frac{k-1}{2} M_\infty^2 F_0'^2,$$
$$\tau_0 = \frac{T_{0W}}{T_\infty} + \left(1 - \frac{T_{0W}}{T_\infty} + \frac{k-1}{2} M_\infty^2\right) F_0' - \frac{k-1}{2} M_0^2 F_0'^2, \tag{19}$$

with T_{0W} denoting the plate temperature and the subscript ∞ referring to parameters of an inviscid fluid. Using these relations, one can obtain solutions for these two cases, namely for a plate either thermally insulated or isothermal up to the point $\xi = 1$.

Initial Plate Segment Thermally Insulated. The solution to the first equation in (9) is adjoined with the first expression in (19) to become

$$P_0 = \left(1 + \frac{k-1}{2} M_\infty^2\right) N. \tag{20}$$

The second term of the second expansion (3) can be expressed as

$$P_1 = P_{10} + C_1 P_{11} + D_1 P_{12}, \tag{21}$$

where $P_{10} = N^2$, $P_{1j} = [aF_0''(0)]^{-1/3} p_{1j}\{[aF_0''(0)]^{1/3}N\}$, $j = 1, 2$. Functions p_{1j} are determined from the expressions

$$p_{1j}''' = t p_{1j}' - t^2 p_{1j}'',$$
$$p_{11}(0) = 0, \quad p_{11}'(0) = 1, \quad p_{11}''(0) = 0; \tag{22}$$
$$p_{12}(0) = 0, \quad p_{12}'(0) = 1, \quad p_{12}''(0) = 1$$

and are shown in Fig. 2a. Constants C_1 and D_1 can be easily determined from the boundary conditions at $N = 0$ and $N \to \infty$.

Initial Plate Segment Isothermal. The solutions to the binomial will be written for the case where the temperature of the initial plate segment is T_{0W}:

$$P_0 = \frac{T_{0W}}{T_\infty} N, \tag{23}$$

$$P_1 = P_{10} + C_1 P_{11} + D_1 P_{12}, \tag{24}$$

where $P_{10} = 1/2 \cdot F_0''(0)\,(1 - T_{0W}/T_\infty + k - 1/2 \cdot M_\infty^2) N^2$, P_{11}, and P_{12} are formally similar to the already found expressions in (6). Constants C_1 and D_1 can also be easily found from specific boundary conditions.

With the aid of these relations we now determine

$$\tau_W = \sqrt{\frac{\rho_\infty \mu_\infty U_\infty^3}{x}}\,[0.332 + \varepsilon 1.367 a^{1/2} A_1] \tag{25}$$

for a plate with a thermally insulated initial segment:

$$\begin{gathered} T_W = T_\infty \left[\left(1 + \frac{k-1}{2} M_\infty^2 \right) + \varepsilon B_1 \right], \\ q_W = \sqrt{\frac{U_\infty \rho_\infty}{\mu_\infty x}}\, \lambda_\infty T_\infty 0.517 B_1, \end{gathered} \tag{26}$$

and for a plate with an isothermal initial segment

$$T_W = T_{0W} + \varepsilon T_\infty B_1,$$

$$q_W = \sqrt{\frac{U_\infty \rho_\infty}{\mu_\infty x}}\, \lambda_\infty T_\infty \left[0.517 B_1 - 0.332 \left(1 - \frac{T_{0W}}{T_\infty} + \frac{k-1}{2} M_\infty^2 \right) \right], \tag{27}$$

where the value of ε obtained from the first equation in (5) with $b_j = 0$ will be

$$\varepsilon = \frac{(x^{3/2} - x_*^{3/2})^{1/3}}{x^{1/2}}. \tag{28}$$

Wake behind a Plate. In order to obtain solutions for the wake region behind a plate, it is convenient to make the following substitutions:

$$G_j = (2a)^{-1/2} g_j [(2a)^{1/2} N], \quad P_j = (2a)^{-1/2} p_j [(2a)^{1/2} N]. \tag{29}$$

Omitting all intermediate calculations, we show here the final results and the values of the essential parameters:

$$k_1 F_0''(0) = 0.3737, \quad k_2 = 0, \quad k_3 = 0, \quad g_1 = 0, \quad g_2 = 0. \tag{30}$$

Solutions g_0 and g_3 are shown in Fig. 2b. While finding the solutions for p_j, we will also consider two different flow modes at the plate.

Thermally Insulated Plate. The solutions for this case are simple:

$$\begin{gathered} p_0 = \left(1 + \frac{k-1}{2} M_\infty^2 \right) t, \quad p_1 = 0, \\ p_2 = -\frac{1}{2}(k-1) M_\infty^2 \int_0^t g_0'^2\, dt, \quad p_3 = 0. \end{gathered} \tag{31}$$

Isothermal Plate. The solutions for this case are found by direct integration of the equations for p_j:

$$\begin{gathered} p_0 = \frac{T_{0W}}{T_\infty} t, \quad p_1 = \left(1 - \frac{T_{0W}}{T_\infty} + \frac{k-1}{2} M_\infty^2 \right) g_0, \\ p_2 = -\frac{1}{2}(k-1) M_\infty^2 \int_0^t g_0'^2\, dt, \quad p_3 = 0. \end{gathered} \tag{32}$$

We now determine the velocity in the wake region along the x-axis:

$$u = U_\infty [\varepsilon\, 0.675 + \varepsilon^4\, 0.053] \tag{33}$$

and the temperature for both cases:

for a thermally insulated plate

$$T = T_\infty \left[\left(1 + \frac{k-1}{2} M_\infty^2 \right) - \varepsilon^2\, 0.457 \; \frac{k-1}{2} M_\infty^2 \right], \tag{34}$$

for an isothermal plate

$$T = T_\infty \left[\frac{T_{0W}}{T_\infty} + \varepsilon\, 0.675 \left(1 - \frac{T_{0W}}{T_\infty} + \frac{k-1}{2} M_\infty^2 \right) - \varepsilon^2\, 0.457 \, \frac{k-1}{2} M_\infty^2 \right]. \tag{35}$$

The dimensionless velocity in the wake along the x-axis, as calculated according to formula (33), is shown in Fig. 3.

NOTATION

μ is the dynamic viscosity;
T is the temperature;
Pr is the Prandtl number;
ψ is the flow function;
Ma is the Mach number for outer stream;
u is the velocity;
ρ is the density;
λ is the thermal conductivity;
k is the adiabatic exponent.

LITERATURE CITED

1. M. VanDyke, Perturbation Methods in Fluid Mechanics [Russian translation], Izd. Mir, Moscow (1967).
2. S. Goldstein, Proc. Cambridge Phil. Soc., 26 (1930).
3. S. Goldstein, Present State of the Art in the Hydrogasdynamics of Viscous Fluids [Russian translation], Izd. Inostr. Lit., Moscow (1948), Vol. 1.
4. E. Truckenbrodt, Abh. Braunschweig. Wissenschaft. Ges. [German], 4 (1952).
5. M. R. Denison and E. Baum, J. AIAA, 1, No. 2 (1963).
6. E. Baum, J. AIAA, 2, No. 1 (1964).
7. Yu. A. Dem'yanov, Zh. Vychisl. Matem. i Matem. Fiz., 7, No. 4 (1969).
8. Yu. A. Dem'yanov, A. N. Pokrovskii, and V. N. Shmanenkov, Mekhan. Zhidk. i Gaza, No. 2 (1971).
9. V. N. Shmanenkov, Mekhan. Zhidk. i Gaza, No. 3 (1966).
10. L. G. Loitsyanskii, Laminar Boundary Layer [in Russian], Fizmatgiz, Moscow (1962).
11. G. Schlichting, Theory of the Boundary Layer [Russian translation], Izd. Nauka, Moscow (1969).